ON A RELATIVE ISODIAMETRIC INEQUALITY FOR CENTRALLY SYMMETRIC, COMPACT, CONVEX SURFACES

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Abstract. We consider the problem of dividing a centrally symmetric, compact, convex surface into two regions in such a way that the intrinsic diameter of both regions is as small as possible. We discuss the best upper bound for the ratio between the area of the smallest region (relative area) and the maximal relative intrinsic diameter. We provide necessary and sufficient conditions for attaining the equality sign. As a consequence from this conditions, there are many surfaces for which equality sign is never attained. We make a complete study of the special case of the cube surface obtaining the best possible upper bound for this surface.

1. Introduction

Relative geometric inequalities regard the division of a given set into two (or more) parts in such a way that some geometric magnitude is maximized or minimized. Historically, the first relative geometric inequalities considered were inequalities for convex subsets of $\mathbb{R}^n$. References about these inequalities are [3],[4], [5], [10].

The problem also makes sense for subdivisions of compact surfaces (see, for instance, [2],[6], [9]).

In this paper we are interested in dividing a centrally symmetric, compact, convex surface into two regions in such a way that the intrinsic diameter of both regions is as small as possible; in these cases, both regions would be as much "rounded" as possible.

[6] presents an inequality providing the best possible general bound. The main goal of the present paper is to discuss when the equality sign is attained. To this end we pay special attention to the centrally symmetric surfaces with **constant antipodal distance**.

In the case of surfaces of revolution we also characterize those for which the equality sign is never attained, and obtain some conditions for the general case. In the surfaces where equality sign is never attained we can ask what would be the subdivision providing the best bound. We give an example of how can we answer this question in the case of the cube surface; the proof that we give for the cube surface shows how a discrete argument can be used to solve a problem that is essentially continuous. In the last section, we apply the characterization that we have obtained for the surfaces of revolution to make a complete discussion of the case of the surface of a solid cylinder.

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2. Preliminaries

Definition 1. *A* **convex body** K *in* $\mathbb{R}^3$ *is a convex compact subset of* $\mathbb{R}^3$.

Definition 2. *A* **convex, compact surface** $S \in \mathbb{R}^3$ *is the boundary of a convex body in* $\mathbb{R}^3$.

Remark 1. *Although not all the points in the boundary of a convex body are regular (there is a unique supporting plane) and not all of them are even smooth, it is well known that the boundary of a convex body has strong differentiability properties:*

1. *For any convex body* K *most* $x \in \partial K$ *are regular.*
2. *For any convex body* K *nearly all* $x \in \partial K$ *are smooth.*
3. *For any proper n-dimensional convex body* K *the set of singular points* $x \in \partial K$ *has* σ*-finite (n-2)-dimensional Hausdorff measure.*
4. *For any two points in the boundary of a convex body* K*, there is at least one geodesic segment connecting them.*

(See, for instance, [7], [8]*).*

So, there is no need to assume any further differentiability assumptions for most of the results that we present in this paper; in the few particular cases in which such assumptions are required, we shall state them explicitly.

Definition 3. *Let* S *be a compact surface. The* **intrinsic distance** *between two points of* S*,* p *and* q *is:*

$$d_i(p,q) = \min_{\alpha}\{L(\alpha), \alpha(a) = p \text{ and } \alpha(b) = q\},$$

where α *is a continuous curve* $\alpha : [a,b] \longrightarrow S$ *joining* p *and* q.

Definition 4. *Let* S *be a compact surface. A* **region** *in* S *is a compact connected subset of* S.

Definition 5. *Let* R *be a region contained in a compact surface* S*. The* **intrinsic diameter** *of* R *is:*

$$D(R) = \max\{d_i(p,q),\ p,q \in R\}.$$

Remark 2. *Usually both notions of intrinsic distance and intrinsic diameter are defined by means of "inf" and "sup" instead of "min" and "max". By standard compactness arguments there is a curve and a pair of points, respectively, for which the extremal value is attained.*

Definition 6. *Let* S *be a compact convex surface and let* α *be a Jordan curve on* S*; by the Jordan curve theorem we know that* α *divides* S *into two connected regions* R *and* $\overline{S \setminus R}$*. We define:*
- the **maximum relative diameter** *of* R *as:*

$$d_M(R,S) = \max\{D(R), D(\overline{S \setminus R})\}$$

- the **relative surface area** *of* R *as:*

$$A(R,S) = \min\{A(R), A(\overline{S \setminus R})\}$$

Definition 7. *Let S be a centrally symmetric compact convex surface. The* **minimal antipodal distance** *is:*

$$\delta_m(S) = \min_{r \in S}\{d_i(r,r'), \text{ where } r \text{ and } r' \text{ are antipodal points in } S\}.$$

We denote by Q the subset of all points in S attaining the minimal antipodal distance:

$$Q := \{q \in S : d_i(q,q') = \delta_m(S)\}.$$

Definition 8. *Let S be a convex, compact surface. A Jordan curve $\alpha : [0, L[\to S$* **bisects** *S if it divides S into two regions R and $\overline{S \setminus R}$ in such a way that $A(R) = A(\overline{S \setminus R}) = A(S)/2$.*

Lemma 1. *Let S be a convex, compact surface. Let α and β be both of them Jordan curves bisecting S. Then, $\alpha \cap \beta \neq \emptyset$.*

Proof. If $\alpha \cap \beta = \emptyset$, both curves would divide S into three regions.

$$A(S) = A(R_\alpha) + A(R_\beta) + A(R_{\alpha\beta}) = \frac{A(S)}{2} + \frac{A(S)}{2} + A(R_{\alpha\beta}) > A(S),$$

and this is a contradiction.

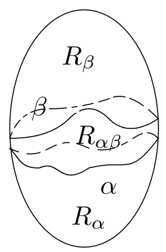

FIGURE 1

□

The following Corollary was already proved in [6], but for the sake of completeness we give the proof again.

Corollary 1. *Let S be a centrally symmetric compact convex surface; if α is a Jordan curve bisecting S into two complementary regions S_1 and S_2, then α contains two antipodal points.*

Proof. From the assumptions of the corollary, we have:

$S = S_1 \cup S_2$,
$\text{int}(S_1) \cap \text{int}(S_2) = \emptyset$,
$A(S_1) = A(S_2) = \frac{1}{2}A(S)$,
$\partial S_1 = \partial S_2 = \alpha$.

To each point p of α we can associate its antipodal point p' with respect to the center of symmetry of S and so we obtain another simple, closed, continuous curve α' (the **antipodal curve** of α) determined by all the points p'. As a consequence of the Lemma 1, α and α' intersect in at least two antipodal points r and r'.

□

Lemma 2. *([11]) Let S be a centrally symmetric, convex, compact surface. Then, there are antipodal points p and p' such that $d_i(p,p') = D(S)$.*

Remark 3. *([11]) If S is a centrally symmetric, convex, compact surface of revolution, with center O, then $D(S) = d_i(p,p')$, where p and p' are the* **poles***. (i.e., the points determined by the intersection of S with the axis of revolution). The* **geodesic segments** *(i.e., minimizing arcs of geodesics) joining p and p' are the meridians. The* **equator** *E is the curve determined by the intersection of S with a plane perpendicular to the axis pp' passing through the center O.*

Proposition 1. *If S is a centrally symmetric, convex, compact surface of revolution, then, $E \subset Q$.*

Proof. If $q, q' \in Q$, let α be the shortest path joining q with q', and let α' be its antipodal path. Obviously $\alpha \cup \alpha'$ bisects S. According to Lemma 1, $\alpha \cup \alpha'$ intersects the equator in e, e'. So, $d_i(e,e') \leq \frac{L(\alpha)+L(\alpha')}{2} = L(\alpha)$. Hence, $e, e' \in Q$. Due to the rotational symmetry, we conclude that $E \subset Q$. □

Proposition 2. *Let R be a region in a centrally symmetric, convex, compact, surface S. Then,*

$$\delta_m(S) \leq d_M(R,S) \leq D(S). \tag{1}$$

Proof. Without loss of generality we can assume that $A(R) \geq A(\overline{S \setminus R})$; then R contains another region R' with exactly half of the area. In the boundary of R' there are at least (according to Corollary 1 two antipodal points q and q'. Obviously, q and q' also belong to R. Then, $\delta_m(S) \leq d_i(q,q') \leq d_M(R,S)$.

On the other hand, $d_M(R,S) = \max\{D(R), D(\overline{S \setminus R})\} \leq D(S)$ as a consequence of the fact that the functional $D(.)$ is monotonous with respect to set inclusion. □

Definition 9. *Let S be a compact surface. Let $r \in S$. A point $s \in S$ is a* **farthest point** *of r if $d_i(r,s) \geq d_i(r,t)$, $\forall t \in S$. We denote by $f(r)$ the set of farthest points of r. If α is a Jordan curve dividing S into two regions R and $\overline{S \setminus R}$, we say that $s \in R$ is a* **farthest point of** *$r \in R$ if $d_i(r,s) \geq d_i(r,t)$, $\forall t \in R$. We denote by $f_R(r)$ the set of all such points.*

3. A relative isodiametric inequality

In [6] the following relative isodiametric inequality was obtained:

Corollary 2. *Let S be a centrally symmetric compact convex surface. Let α be a Jordan curve on S dividing S into two complementary regions, R and $\overline{S \setminus R}$. Then,*

$$\frac{A(R,S)}{d_M(R,S)^2} \leq \frac{A(S)}{2\delta_m(S)^2}, \tag{2}$$

where $\delta_m(S)$ is the minimal antipodal distance of S.

Proof. As a consequence of Proposition 2, $\delta_m(S) \leq d_M(R,S)$. On the other hand, clearly, $A(R,S) \leq A(S)/2$ and the assertion follows. □

Proposition 3. *If S is a centrally symmetric, compact, convex surface with constant antipodal distance, equality sign in* (2) *is attained for any division of S into two regions with the same area, whose common boundary is a Jordan curve.*

Proof. As $A(R,S) \leq \frac{A(S)}{2}$, to attain equality sign in (2) is necessary that $A(R) = \frac{A(S)}{2}$. As S has constant antipodal distance, according to (1):

$$\delta_m(S) = D(S) = d_M(R,S).$$

Hence, equality sign is attained. □

Remark 4. *Let S be a compact surface; let $r, s \in S$. We denote by γ_{rs} the* **geodesic segment** *joining r and s.*

Proposition 4. *If S is a centrally symmetric, compact, convex surface without constant antipodal distance, there is at least a bisection of S for which equality sign is not attained in* (2).

Proof. Let p and p' be two antipodal points in S such that $D(S) = d_i(p,p')$ (see Lemma 2).

Let $\gamma_{pp'}$ be a geodesic segment joining p and p' and let $\gamma'_{pp'}$ be its antipodal curve ($\gamma'_{pp'} = \gamma_{p'p}$). Let $\alpha = \gamma_{pp'} \cup \gamma'_{pp'}$. α is a Jordan curve bisecting S into two symmetric regions R and $\overline{S \setminus R}$. Obviously, $A(R) = A(\overline{S \setminus R})$, and $p, p' \in R \cap \overline{S \setminus R}$.

As we are assuming that S has not constant antipodal distance, $D(S) > \delta_m(S)$. So, $d_M(R,S) = d_i(p,p') = D(S) > \delta_m(S)$, and equality is not attained for this bisection. □

Remark 5. *If the centrally symmetric, convex, compact surface S is also differentiable, we can guarantee that for any point $p \in S$ such that $\delta_m(S) = d_i(p,p')$, the Jordan curve $\alpha = \gamma_{pp'} \cup \gamma_{p'p}$ is a closed geodesic.*

There are many examples of centrally symmetric, compact, convex surfaces with constant antipodal distance: the sphere, the double disk, the symmetric lens, and many others (see, for instance, [6]). In [11], Vîlcu provides a large family of surfaces of revolution of constant antipodal distance, including the ellipsoid: $E = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, b < a\}$. All the examples of surfaces with constant antipodal distance that we know are surfaces of revolution. It would be interesting to know if there is any example which is not a surface of revolution.

There are also examples of surfaces without constant antipodal distance for which equality sign is attained for some subdivisions but not for all of them. One example would be the surface of the cap body $K := conv\{\overline{B}((0,0,0),1000) \cup (0,0,1001) \cup (0,0,-1001)\}$.

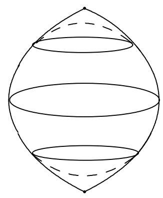

FIGURE 2

As this surface has not constant antipodal distance, according to Proposition 4, there is at least a bisection of this surface for which equality sign is not attained. On the other hand, if we divide this surface by the "equator" $\beta(t) = (1000\cos t, 1000\sin t, 0)$, $t \in [0, 2\Pi[$, the equality sign is attained at least for this particular subdivision.

So a natural problem is to characterize the centrally symmetric, compact, convex surfaces for which equality sign in (2) is never attained. We shall discuss this problem for the surfaces of revolution in the next section. If this is not the case, we can at least obtain a necessary and a sufficient condition; we shall state both of them also at the end of the next section.

4. Surfaces of revolution

Proposition 5. *Let S be a centrally symmetric, compact, convex surface and let α be a Jordan curve dividing S into two regions R and $\overline{S \setminus R}$. There is at least a point $r \in \alpha$ such that $D(R) = d_i(r,s)$, where $s \in f_R(r)$.*

Proof. Let us assume that both endpoints of the intrinsic diameter of R belong to the interior of R. Let us consider the closed geodesic ball $\overline{B(s, D(R))}$. This ball cannot be contained in the interior of R; if that were the case $d_i(s,q) > D(S)$, $\forall q \in \alpha$, and then, the intrinsic diameter would not be $D(R)$. By continuity $\partial B(s, D(R)) \cap \alpha \neq \emptyset$, so there must be a point $r \in \alpha$ such that $D(R) = d_i(r,s)$.

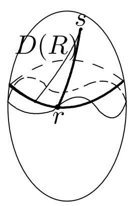

FIGURE 3

□

Definition 10. *If S is a centrally symmetric, convex, compact surface of revolution with equator E, we define* $D_E(S) = \max\{d_i(e, s) : e \in E, s \in S\}$.

Proposition 6. *Let S be a centrally symmetric, compact, convex surface of revolution, and let α be a Jordan curve dividing S into two regions R and $\overline{S \setminus R}$. Then,*

$$d_M(R, S) \geq D_E(S) \geq \delta_m(S), \tag{3}$$

and equality in the left inequality is attained if α is the equator.

Proof. For the left inequality we distinguish two cases:

If α intersects the equator in a point e then,

$$d_M(R, S) \geq \max\{d_i(e, s) : e \in E, s \in S\} = D_E(S).$$

If α does not intersect the equator, the bigger region contains the equator and suitable farthest points, and hence $d_M(R, S) \geq D_E(S)$.

Since $E \subset Q$ we have $\delta_m(S) = d_i(e, e')$, for any point $e \in E$, and the right inequality follows from the definition of $D_E(S)$.

For the equality case,let R and R' be the regions defined by the equator. The intrinsic diameter of R has at least one endpoint in the boundary and from the definition of $D_E(S)$ it follows $D(R) = D_E(S)$. □

Proposition 7. *Let S be a centrally symmetric, compact, convex surface of revolution. Equality can be attained in* (2) *if and only of*

$$D_E(S) = \delta_m(S).$$

Proof. In the case of equality we have $d_M(R, S) = \delta_m(S)$ for a suitable region R. From (3) it follows $d_M(R, S) = D_E(S) = \delta_m(S)$.

On the other hand, if $D_E(S) = \delta_m(S)$, then the region R bounded by the equator satisfies $A(R, S) = A(S)/2$ and $d_M(R, S) = D_E(S) = \delta_m(S)$ and the equality in (2) follows. □

In section 6, we shall show an application of Proposition 7 to the case of the surface of a solid cylinder.

Remark 6. *It is easy to see that the characterization given in Proposition 7 of those centrally symmetric, convex, compact surfaces of revolution for which equality sign is never attained in* (2) *is equivalent to the following two statements:*

1) *For any point e in the "equator" of S, there is a closed geodesic ball centered at e whose complementary is not connected.*
2) *For any point e in the "equator" of S, the set of farthest points $f(e)$ has more than one element.*

We can extend some of the arguments used above to the general case of centrally symmetric, convex, compact surfaces in the following way:

Proposition 8. *Let S be centrally symmetric, convex, compact surface without constant antipodal distance. If for any $q \in Q$, $d_i(q, r) > \delta_m(S)$, for any $r \in f_R(q)$, then equality is never attained in* (2).

Proof. Equality in (2) would require that α bisects S. Let us consider any pair of antipodal points $q, q' \in Q$; let $\beta = \gamma_{qq'} \cup \gamma_{q'q}$; β also bisects S. As a consequence of Lemma 1, $\alpha \cap \beta \neq \emptyset$, so α contains at least one point $q \in Q$. As we are assuming that $d_i(q, r) > \delta_m(S)$ for any $r \in f_R(q)$, then equality is never attained in (2). □

Proposition 9. *Let S be a centrally symmetric, convex, compact surface without constant antipodal distance. If equality is never attained in* (2)*, there is at least one point $q \in Q$ such that $d_i(q, r) > \delta_m(S)$, for any $r \in f(q)$.*

Proof. If λ is a Jordan curve dividing S into two regions R' and $\overline{S \setminus R'}$, let R' be the region such that $A(R') \geq A(\overline{S \setminus R'})$. R' contains a smaller region R such that $A(R) = A(S)/2$. R is bounded by a Jordan curve α bisecting S. By the argument used in the proof of Proposition 8, we know that α contains at least one point $q \in Q$. Let $\beta = \gamma_{qq'} \cup \gamma_{q'q}$. By our assumption that equality is never attained in (2), we know that equality is not attained by β; so there is at least one point $q \in \beta$ such that $d_i(q, r) > \delta_m(S)$ for any $r \in f_R(q)$: as β is a self-antipodal curve, both regions R_β and $\overline{S \setminus R_\beta}$ bounded by β are congruent, and according to Proposition 5, $d_i(q, r) = D(R) > \delta_m(S)$, for any $r \in f(q)$. □

5. The cube surface

The cube surface C is an example of a centrally symmetric, convex, compact surface for which equality is never attained in (2). We are going to show which is the best bound that we can attain in this case.

In the following we understand:

(1) the $(0, 0, 1)$-half cube as the intersection of C with the half-space $\{z \geq \frac{1}{2}\}$; this half-space is bounded by a plane whose normal vector is $(0, 0, 1)$ (Fig. 4(a)).
(2) the $(1, 1, 0)$-half cube as the intersection of C with the half space $\{x + y \leq 1\}$; this half space is bounded by a plane whose normal vector is $(1, 1, 0)$ (Fig. 4(b)).

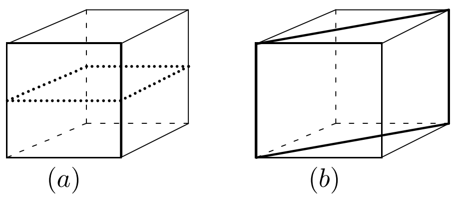

Figure 4

For two points a, b in the cube there is always at least one shortest path γ representing the intrinsic distance $d_i(a, b)$. It is obvious that γ is a union of line segments each of them contained in one face. In the following we consider all possible types of such shortest paths. We distinguish three cases:

1) If a and b are in the same face then γ is necessarily $\overline{ab}$.

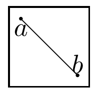

Figure 5

2) If a and b are contained in adjacent faces f_a, f_b then there are two options:

a) γ crosses only the edge separating f_a and f_b.
b) γ passes through one of the two faces which are adjacent to f_a as well as f_b.

Figure 6

3) If a and b are contained in opposite faces f_a, f_b, then γ has to pass through at least one of the four faces between f_a and f_b. Since γ can pass through at most two (adjacent) faces among these four, we have only two possible different situations:

Figure 7

Summarizing all the different cases we obtain:

Lemma 3. *Let a, b be two points in C. Then for any shortest path γ from a to b there is an unfolding of the cube such that its image γ' is a line segment.*

Proposition 10. *The intrinsic diameter of the $(0,0,1)$–half cube is $d_0 = \sqrt{145}/6$.*

Proof. From Proposition 5 it follows that the diameter is attained for a point X in the boundary $z = 1/2$ of the half cube and another point Y. If Y has z–coordinates $< 2/3$ then $d_i(X,Y) < \sqrt{2^2 + (1/6)^2} = d_0$. So we can assume that Y has z–coordinate $\geq 2/3$. If Y is contained in the top face or an adjacent half face then the intrinsic distance to X is at most $\sqrt{(3/2)^2 + (4/3)^2} = d_0$. So we can assume that Y is contained in the shaded area of the opposite half face (see figure 8). Let $x \in [0, 1/2]$ be the distance of X to its nearest vertex and let X', X'' be the points equivalent to X as indicated in Figure 8.

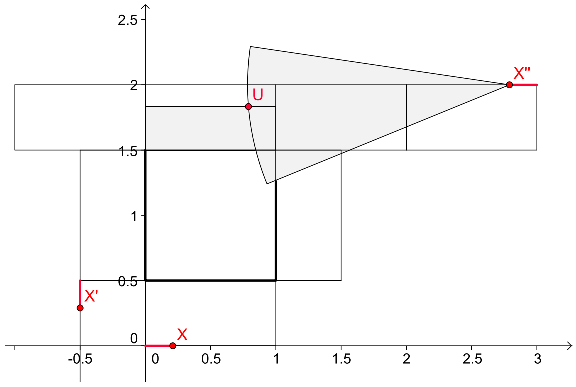

FIGURE 8

In the set of all points of the shaded rectangle with distance $\geq d_0$ to X'' the point U (see figure 8) has maximal distance to X as well as to X'. These distances are

$$d(X,U) = \sqrt{(1-2x)^2 + (11/6)^2} = \sqrt{156/36 + 4x^2 - 4x}$$

and

$$d(X',U) = \sqrt{(3/2-x)^2 + (4/3+x)^2} = \sqrt{145/36 - x/3 + 2x^2},$$

respectively. For $0 \leq x \leq 1/6$ we have $d(X,U) \leq d_0$ and for $1/6 \leq x$ we have $d(X',U) \leq d_0$.

It follows that the intrinsic diameter of the half cube is at most d_0. On the other hand it is easy to check that for the case $x = 0$ there is no path shorter than d_0.

□

Lemma 4. *Let α be a Jordan curve in C dividing C into two regions R and $\overline{C \setminus R}$. Then*

$$d_M(R,C) \geq d_0 := \frac{\sqrt{145}}{6} \cong 2.0069\ldots$$

Proof. We assign red color to all the points in R and blue color to all the points in $\overline{C \setminus R}$ (We assign both colors to all points in α).

We consider three types of special points:

1) the vertices of C,
2) the centers of the edges of C,
3) the inner sixth parts of edges of C
4) the outer sixth parts of edges of C

If $d_M(R,C) < d_0$ then the following points must have different colors (because they have distance $\geq d_0$):

(a) center + opposite inner sixth parts

(b) inner sixth part + opposite inner sixth part
(c) inner sixth part + opposite outer sixth part
(d) outer sixth part + opposite vertex
(e) vertex + opposite vertex

Let the center of an edge E be blue. Then the two inner sixth parts of the opposite edge E' are only red by (a); then the two inner sixth parts of E by (b) as well as the outer sixth parts of E by (c) are only blue. Analogously, the center and the two outer sixth parts of E' are only red by (d).

Consequently, all "our points" on one edge necessarily have the same color. So the adjacent 4 edges to our blue edge have to be blue; in the next step 6 further edges have to be blue and finally the remaining edge and so all twelve edges are blue. This is a contradiction to (e).

Thus $d_M(R, C) \geq d_0$.

On the other hand if α is the intersection of C with a plane parallel to and at a distance $\frac{1}{2}$ of two opposite faces of C then $d_M(R, C) = d_0$.

□

Remark 7. *If α is a Jordan curve in C dividing C into two regions R and $\overline{C \setminus R}$, as an immediate consequence of Corollary 1 we have that $A(R, C)/d_M(R, C)^2 \leq 3/4 = 0.75$; however this is not the best upper bound.*

Proposition 11. *Let α be a Jordan curve in C dividing C into two regions R and $\overline{C \setminus R}$. Then*

$$0 \leq \frac{A(R, C)}{d_M(R, C)^2} \leq \frac{108}{145} \cong 0,74482...,$$

and both are the best possible bounds.

Proof. For the lower bound we consider R as a region with empty interior, for instance, a straight line segment contained in C.

The upper bound follows from the following facts:
a) that the greatest possible value for $A(R, C)$ is 3,
b) that according to Proposition 8 the smallest possible value for $d_M(R, C)$ is $\sqrt{\frac{145}{36}}$.

So,

$$\frac{A(R, C)}{d_M(R, C)^2} \leq \frac{3}{\frac{145}{36}} = \frac{108}{145}.$$

This inequality is tight since both bounds in a) and b) are attained for the (0,0,1)–cube.

□

Remark 8. *If we consider the particular case of bisections then an immediate consequence of $d_M(R, C) \leq D(C) = \sqrt{5}$ is that*

$$\frac{3}{5} \leq \frac{A(R, C)}{d_M(R, C)^2} \leq \frac{108}{145}.$$

The lower bound is attained for many regions R, for instance the $(1, 1, 0)$-half cube.

6. The surface of a solid cylinder

Let K be the solid circular cylinder with radius R and altitude h: $K := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq R^2 \text{ and } -\frac{h}{2} \leq z \leq \frac{h}{2}\}$. Let $S = \partial K$ be its surface.

Let $p = (0,0,h/2)$ and $p' = (0,0,-h/2)$ be the poles of S. According to Remark 3, $D(S) = d_i(p,p') = 2R+h$.

Let us consider the following antipodal points in the equator E: $e = (-R,0,0)$ and $e' = (R,0,0)$. The arc of equator joining e and e' is a geodesic with length ΠR.

We distinguish two cases: if $h \leq (\Pi - 2)R$, then S has constant antipodal distance, and equality in 2 is attained for any bisection of S; if $h > (\Pi-2)R$, then $D_E(S) > \delta_m(S)$, and equality is never attained in (2).

To prove this, we use the following Lemma:

Lemma 5. *The shortest path joining e and e' passing through the upper disc has length* $\min(h+2R, \sqrt{h^2+R^2\pi^2})$.

Proof. Let γ be a curve joining e and e' passing through the upper disc with minimal length. Then it consists of three arcs (an arc of cylindrical helix joining e with a point r in the circumference of the upper disc, a straight line segment joining r with another point s in the same circumference, and an arc of cylindrical helix joining s with e'). Unfolding the lateral surface of the cylinder, the arcs of cylindrical helix er and se' are transformed in straight line segments in the rectangle with base πR and altitude $h/2$.

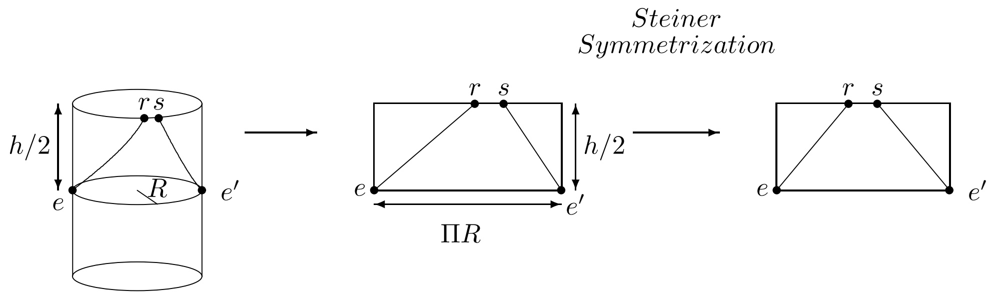

Figure 9

Using Steiner symmetrization in this rectangle, we can conclude that the shortest value for the length is attained if $L(er) = L(se')$. So the total length of this curve is $f(x) = \sqrt{h^2+R^2\cdot(\pi-x)^2} + 2R\sin(x/2)$, where x is the angle of the disc segment given by r and s.

The derivative $f'(x) = R^2(x-\pi)/\sqrt{h^2+R^2\cdot(\pi-x)^2} + \cos(x/2)$ has a zero in $x=\pi$ and possibly another one in $x = x^*$, where x^* is the solution of the equation $\tan(x/2)(\pi - x) = h/R$ (it exists in the case $h \leq 2R$).

If x^* exists, then a consequence of $f'(0) > 0$ is that f can have in x^* only a local maximum. In any case we have $\min\{f(x) : 0 \leq x \leq \pi\} = \min\{f(0), f(\pi)\}$ and the assertion follows. ☐

Now, we conclude with the classification of cylinders:

Proposition 12. *If $h \leq (\pi - 2)R$, then S has constant antipodal distance.*

Proof. As we know (Proposition 1) all points in the equator belong to Q and hence $\delta_m(S) = d_i(e, e')$. So it suffices to prove that $d_i(e, e') \geq D(S) = d_i(p, p') = h + 2R$.

Let γ be a curve joining e and e'. If γ passes through the upper disc, then Lemma 4 guarantees that $L(\gamma) \geq \min(h + 2R, \sqrt{h^2 + R^2\pi^2}) = h + 2R$ by our assumption on h. If γ is completely contained in the lateral surface of K, it has length at least πR (to see this we use the isometry determined by the unfolding of the lateral surface in a rectangle).

Hence $d_i(e, e') \geq \min(h + 2R, \pi R) = h + 2R$. □

Proposition 13. *If $h > (\pi - 2)R$, then $D_E(S) > \delta_m(S)$, and equality is never attained in (2).*

Proof. Let $e = (-R, 0, 0)$ and $e' = (R, 0, 0)$. By our assumption on h we can choose an $\epsilon > 0$ such that $\sqrt{\pi^2 R^2 + \epsilon^2} < \min(h + 2R, \sqrt{h^2 + R^2\pi^2}) - \epsilon$.

Now let $r = (R, 0, \epsilon)$ and let γ be a path in S joining e and r. If γ is completely contained in the lateral surface of K, then $L(\gamma) \geq \sqrt{\pi^2 R^2 + \epsilon^2}$. If γ passes through the upper disc, then the concatenation γ' of γ and the segment $e'r$ satisfies the condition of Lemma 4 and so $L(\gamma) = L(\gamma') - \epsilon \geq \min(h + 2R, \sqrt{h^2 + R^2\pi^2}) - \epsilon$. By the choice of ϵ we conclude that $D_E(S) \geq d_i(e, r) \geq \sqrt{\pi^2 R^2 + \epsilon^2} > \pi R = \delta_m(S)$.

As a consequence of Proposition 7 equality is never attained in (2). □

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