RELATIVE GEOMETRIC INEQUALITIES FOR COMPACT, CONVEX SURFACES

A. Cerdán[1], C. Miori[2], U. Schnell[3] and S. Segura Gomis[4]

[1,2,4]Departamento de Análisis Matemático, Universidad de Alicante, Campus de San Vicente del Raspeig, E-03080-Alicante, Spain
[3] Fachbereich Mathematik/Naturwissenschatten, University of Applied Sciences Zittau/Görlitz, 02763 Zittau, Germany

e-mail: [1]aacs@alu.ua.es, [2]cm4@alu.ua.es, [3]uschnell@hs-zigr.de, [4]Salvador.Segura@ua.es

Abstract. We obtain relative geometric inequalities for compact, convex surfaces. In particular, we present several inequalities comparing the relative area and the relative perimeter with the maximal and minimal relative diameter. Besides considering the general problem we also consider particular cases where we can obtain sharper results: 1) the so called fencing problems in which only subdivisions into two regions of the same area are considered, 2) the subdivisions obtained by planar cuts.

1. Introduction

Relative geometric inequalities regard the division of a given set G into two parts in a way that some geometric measure is maximized or minimized. Historically the first relative geometric inequalities considered were for convex subsets G of the Euclidean space. References about these inequalities are [4], [5],[6], [7], [13].

The problem also makes sense for subdivisions of compact surfaces. As Osserman [12] pointed out the classical isoperimetric inequality on the sphere obtained by Bernstein [1] is a relative geometric inequality, because any Jordan curve on the sphere, divides it into two regions.

The aim of this paper is to obtain relative geometric inequalities for compact, convex surfaces. In particular, we shall present several relative geometric inequalities comparing the relative area, the relative perimeter and the maximal and minimal relative diameter. Besides considering the general case, we shall also consider particular cases where we can obtain sharper results.

It is sometimes interesting to consider particular versions of the relative geometric problems:

1) the so called fencing problems in which only subdivisions into two regions with the same area are considered,
2) the subdivisions obtained by planar cuts.

Though most of the results hold for any dimension we state them only for three–dimensional surfaces.

Definition 1. *Let S be a compact surface. The* **intrinsic distance** *between two points of S, p and q is:*

$$d_i(p,q) = \min_{\alpha}\{L(\alpha), \alpha(a) = p \text{ and } \alpha(b) = q\},$$

where α is a absolutely continuous curve $\alpha : [a,b] \longrightarrow S$ and L denotes the length of a curve.

Definition 2. *Let S be a compact surface. A* **region** *in S is a compact connected subset of S.*

Definition 3. *Let R be a region contained in a compact surface S. The* **diameter** *of R is:*

$$D(R) = \max\{d_i(p,q),\ p,q \in R\}.$$

Remark 1. *Usually both notions of intrinsic distance and diameter are defined by means of "inf" and "sup" instead of "min" and "max". By standard compactness arguments there is a curve and a pair of points, respectively, for which the extremal value is attained.*

Definition 4. *Let S be a compact convex surface and let α be a Jordan curve on S; by the Jordan curve theorem we know that α divides S into two connected regions R and $\overline{S \setminus R}$. We define:*
- *The* **maximum relative diameter** *of R as:*

$$d_M(R,S) = \max\{D(R), D(\overline{S \setminus R})\},$$

- *The* **minimum relative diameter** *of R as:*

$$d_m(R,S) = \min\{D(R), D(\overline{S \setminus R})\},$$

- *The* **relative surface area** *of S as:*

$$A(R,S) = \min\{A(R), A(\overline{S \setminus R})\} \text{ and}$$

- *The* **relative perimeter** *of R, $P(R,S)$, as the length of the curve α.*

Obviously,

$$d_m(R,S) \leq d_M(R,S) \leq D(S).$$

Definition 5. *Let S be a centrally symmetric, compact surface. The* **minimal antipodal distance** *is:*

$$\delta_m(S) = \min\{d_i(p,p') \text{ where } p \text{ and } p' \text{ are antipodal points in } S\}.$$

The following lemma extends a remark that Bernstein [1] made for the sphere to all centrally symmetric compact convex surfaces.

Lemma 1. *Let S be a centrally symmetric compact convex surface; if α is a Jordan curve (i.e. a simple closed continuous curve) on S dividing S into two complementary regions S_1 and S_2 of equal area, then α contains two antipodal points.*

Proof. From the assumptions of the lemma, we have:

$S = S_1 \cup S_2$,
$\mathrm{int}(S_1) \cap \mathrm{int}(S_2) = \emptyset$,
$A(S_1) = A(S_2) = \frac{1}{2}A(S)$,
$\partial S_1 = \partial S_2 = \alpha$.

To each point p of α we can associate its antipodal point p' with respect to the center of symmetry of S and so we obtain another simple, closed, continuous curve α' (the antipodal curve of α) determined by all the points p'. The curves α and α' intersect: as S is convex it is homeomorphic to the sphere, and so if $\alpha \cap \alpha' = \emptyset$ the Jordan Curve Theorem would imply that S would be divided into three disjoint regions: the first one S_1 bounded by α with $A(S_1) = A(S)/2$, the second one $S_2 = S_1'$ bounded by α' with also $A(S_1') = A(S)/2$, and the third one S_3 between α and α' with strictly positive area: so we obtain a contradiction, and α and α' intersect in at least two antipodal points q and q'.

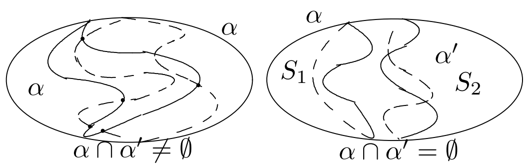

FIGURE 1

□

We shall often use the properties of geodesics ([10]), which are the curves that minimize the intrinsic distance:

Theorem 1. *([2], [9]) (Hopf-Rinow's Theorem) If a length-metric space (M, d) is complete and locally compact, then any two points in M can be connected by a minimizing geodesic and any bounded closed set in M is compact.*

Lemma 2. *Let S be a centrally symmetric compact convex surface. Then there are antipodal points p and p' such that $d_i(p, p') = D(S)$.*

Proof. There are two points p, q with $d_i(p, q) = D(S)$. Let κ_p and κ_q be the shortest path from p to p' and from q to q', respectively. The endpoint p' of κ_p is the starting point of its antipodal path κ'_p. So we can define the sum $\kappa = \kappa_p + \kappa'_p$ which is just their concatenation. Then κ is a closed curve and divides S into at least two parts. If q is contained in κ then $d_i(p, q) \leq d_i(p, p')$ and we are finished. Hence, q and q' are contained in different components and so $\lambda = \kappa_q + \kappa'_q$ and κ intersect in two antipodal points x and x'.

Now we define:

$\alpha =$ path from p to x along κ_p,
$\beta =$ path from p to x' along κ'_p,
$\gamma =$ path from x to q along κ_q,
$\delta =$ path from x' to q along κ'_q.

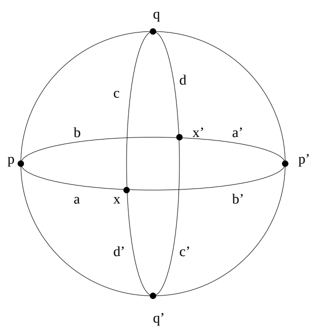

FIGURE 2

By the triangle inequality it follows $d_i(p,q) \le L(\alpha+\gamma)$ and $d_i(p,q) \le L(\beta+\delta)$. Hence, by symmetry,

$$\begin{aligned} D(S) &= d_i(p,q) \le \frac{L(\alpha+\gamma)+L(\beta+\delta)}{2} = \frac{L(\alpha+\beta)+L(\gamma+\delta)}{2} = \frac{L(\kappa_p)+L(\kappa_q)}{2} \\ &= \frac{d_i(p,p')+d_i(q,q')}{2} \le \max\{d_i(p,p'), d_i(q,q')\}. \end{aligned}$$

Without loss of generality let $d_i(p,p') \ge d_i(q,q')$. Then $D(S) \le d_i(p,p')$ and from $d_i(p,p') \le D(S)$ it follows that $D(S) = d_i(p,p')$. □

In the particular case of surfaces of revolution, $D(S) = d_i(p,p')$ where p and p' are the points determined by the intersection of S with the axis of revolution.

Lemma 3. *Let S be a compact convex surface of revolution around the axis pp' where p, $p' \in S$. Then $D(S) = d_i(p,p')$.*

Proof. Obviously, $d_i(p,p) \le D(S)$.

Now let a, b be arbitrary points on S. Take a minimizing geodesic joining p and p' and rotate it twice to get two minimizing geodesics α and β containing a and b respectively.

We define:

$\alpha_1 =$ path from p to a along α,
$\alpha_2 =$ path from p' to a along α,
$\beta_1 =$ path from p to b along β,
$\beta_2 =$ path from p' to b along β,

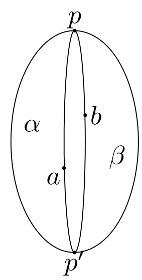

FIGURE 3

By the triangle inequality it follows $d_i(a,b) \le L(\alpha_1+\beta_1)$ and $d_i(a,b) \le L(\alpha_2+\beta_2)$.

Then,

$$d_i(a,b) \le \frac{L(\alpha_1+\beta_1)+L(\alpha_2+\beta_2)}{2} = \frac{L(\alpha_1+\alpha_2)+L(\beta_1+\beta_2)}{2} = \frac{L(\alpha)+L(\beta)}{2} = d_i(p,p').$$

□

Remark 2. *$D(S)$ is the length of the "generating curve" (meridian).*

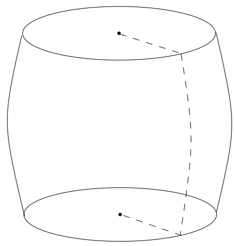

FIGURE 4

2. Maximum Relative Diameter

Proposition 1. *Let S be a compact convex surface. Further let R and $\overline{S \setminus R}$ be complementary regions in S. Then*

$$\frac{A(R,S)}{d_M(R,S)^2} \geq 0$$

and the bound is the best possible.

Proof. There is an extreme point $p \in S$ and a supporting plane Π_0 with $\Pi_0 \cap S = \{p\}$.(See [15]).

We can choose a sequence Π_i of planes parallel to Π_0 with $\Pi_i \to \Pi_0$ whose intersections with S determine a sequence of regions, R_i, such that $\lim_{i\to\infty} A(R_i, S) = 0$. Moreover, $d_M(R_i, S) \to D(S)$ when $i \to \infty$. Then,

$$\lim_{i\to\infty} \frac{A(R_i,S)}{d_M(R_i,S)^2} = 0.$$

□

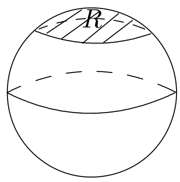

FIGURE 5

If we consider the fencing problem case, we obtain the following result:

Proposition 2. *Let S be a compact convex surface. Further let R and $\overline{S \setminus R}$ be complementary regions in S with equal area. Then,*

$$\frac{A(R,S)}{d_M(R,S)^2} \geq \frac{A(S)}{2D(S)^2},$$

and the inequality is tight.

Proof. We have $A(R,S) = A(S)/2$ and $d_M(R,S) \le D(S)$. So,

$$\frac{A(R,S)}{d_M(R,S)^2} \ge \frac{A(S)}{2D(S)^2}.$$

There are points p and q such that $d_i(p,q) = D(S)$. Take any plane Π containing p and q. It divides the surface into two parts with measure A_1 and A_2. If $A_1 = A_2$ then we are finished. Else we can assume that $A_1 < A_2$. Now we rotate Π around the axes defined by p and q. The areas $A_1(\varphi), A_2(\varphi)$ are changing continuously. Further $A_1(180) = A_2(0)$ and $A_2(180) = A_1(0)$ and so $A_2(180) > A_1(180)$. By continuity, there is an angle φ such that $A_1(\varphi) = A_2(\varphi)$. Since p and q are contained in $A_1(\varphi)$ as well as $A_2(\varphi)$ we have $d_M(R,S) = d_m(R,S) = D(S)$. □

Proposition 3. *Let S be a centrally symmetric compact convex surface. Further let R and $\overline{S \setminus R}$ be complementary regions in S. Then,*

$$\frac{A(R,S)}{d_M(R,S)^2} \le \frac{A(S)}{2\delta_m(S)^2},$$

where $\delta_m(S)$ is the minimal antipodal distance of S.

Proof. Without loss of generality we can assume that $\overline{S \setminus R}$ is the region with greater area; then $\overline{S \setminus R}$ contains another region R' with exactly half of the area. As a consequence of Lemma 1, there are two antipodal points in the boundary of R', x and x'. On the other hand, clearly, $A(R,S) \le A(S)/2$ and the assertion follows. □

This bound is attained by some surfaces like, for instance, the sphere. (Fig. 6).

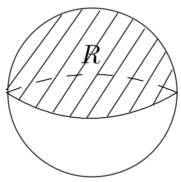

FIGURE 6

However, there are many surfaces for which this bound is not attained:

Example 1. *Let $C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \le 1/100,\ |z| \le 10\}$ and let $S = \partial C$.*

Computing the minimal antipodal distance we obtain that $\delta_m(S) = \frac{\pi}{10}$.
We divide S into two subsets R and $\overline{S \setminus R}$. We can assume without loss of generality that $p \in R$.
We distinguish two cases:

(1) $p' \in R$. In this case $d_M(R,S) = D(S) = 1/5 + 20 > 1/10 + 10$.
(2) $p' \notin R$. If there is a point $q \in \alpha$ such that its z-coordinate is 0, then $d_M(R,S) \ge d_i(p,q) = 1/10 + 10$; if there is not such a point q, then $d_M(R,S)$ is even greater.

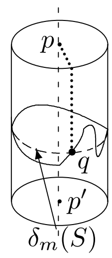

FIGURE 7

In both cases, (1) and (2), $d_M(R,S) > \pi/10$, and so the bound is never attained.

Under the special assumption that the distance between any pair of antipodal points is constant, we can rewrite the bound in Proposition 3 in terms of $D(S)$, and in this case we can also guarantee that the bound is always attained:

Corollary 1. *Let S be a centrally symmetric compact convex surface such that the distance between any pair of antipodal points is constant. Further let R and $\overline{S \setminus R}$ be complementary regions in S. Then,*

$$\frac{A(R,S)}{d_M(R,S)^2} \leq \frac{A(S)}{2D(S)^2}$$

and the bound is the best possible.

Proof. The proof is an immediate consequence of Proposition 3, and the fact that now $\delta_m(S) = D(S)$.

The bound is attained: There are points p, q on S, such that $d_i(p, q') = D(S)$. As in the proof of Proposition 2 there is a plane Π passing through p and q which divides S into two parts R and R' with area $A(S)/2$ and diameter $d(R) = d(R') = D(S)$ and so we have equality. $\square$

There are several interesting examples of compact, convex, centrally symmetric surfaces such that the distance between any pair of antipodal points is constant: the sphere, the double disc. Now we are going to present another example that includes both the sphere and the double disc as particular cases: the symmetric lens and the symmetric segment.

Definition 6. *Let S^2 be the unit sphere and let Π be a plane intersecting S^2; let $C = S^2 \cap \Pi$, and let M^2 be the smallest region of S^2 bounded by C; let M'^2 be the region obtained from M^2 by a symmetry with respect to Π. $L^2 := M^2 \cup M'^2$ is called a symmetric lens. Obviously L^2 is a convex, compact, centrally symmetric surface of revolution around the axis pp', where $p \in M^2$, $p' \in M'^2$ and the segment pp' is orthogonal to Π. We denote by θ the angle between the axis of the sphere S^2 orthogonal to Π and the straight line segment joining the center of the sphere O with an arbitrary point of C.*

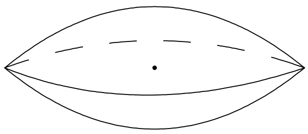

FIGURE 8

Proposition 4. *Let $L^2 \subset \mathbb{R}^3$ be the symmetric lens. Then all antipodal points are at the same distance.*

Proof. Let x and x' be two antipodal points of L^2. If they belong to C, then both points are in S^2, so, obviously, the curve that minimizes the distance between x and x' is the arc of meridian with length 2θ.

So, let us suppose that x and x' do not belong to C. The intersection of the plane through x, x', p, p' gives a path from x to x' with length 2θ. Hence $d_i(x, x') \leq 2\theta$. Let γ be the shortest curve joining x and x'; then $d_i(x, x') = L(\gamma)$. γ intersects C in at least one point b. Let γ' be the antipodal curve of γ, and let b' be the antipodal point of b, which, certainly, belongs to C. So $d_i(b, b') = 2\theta$.

Let us denote by

- γ_1 the arc of γ from x to b
- γ_2 the arc of γ from b to x'
- γ_1' the arc of γ' from b' to x'
- γ_2' the arc of γ' from x to b'.

So,

$$L(\gamma) = L(\gamma_1) + L(\gamma_2) = L(\gamma_1) + L(\gamma_2') \geq d_i(b, b') = 2\theta.$$

□

As an immediate consequence from Proposition 4 and Lemma 2 we have:

Corollary 2. *The diameter of the symmetric lens is the intrinsic distance between any pair of antipodal points.*

Definition 7. *Let $\mathbb{B}^3 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ be the unit ball. Let $\mathbb{K}(s_0) := \{(x, y, z) \in \mathbb{B}^3 : |z| \leq s_0$ where s_0 is a constant such that $0 \leq s_0 \leq 1\}$; $\mathbb{S}(s_0) := \partial\mathbb{K}(s_0)$ is called a* **symmetric segment** *of the sphere (Fig. 9).*

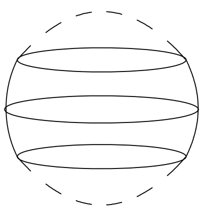

FIGURE 9

Proposition 5. *Let $\mathbb{S}(s_0)$ be a symmetric segment of the sphere. Then all antipodal points are at the same distance.*

Proof. The proof is similar to that of Proposition 4, although the argument has to be accommodated. □

Now, we are going to compare the maximum relative diameter with the relative perimeter:

Proposition 6. *Let S be a compact convex surface and let R and $\overline{S \setminus R}$ be complementary regions in S. Then,*

$$\frac{d_M(R,S)}{P(R,S)} \geq 0$$

and the bound is tight.

Proof. It is sufficient to consider simple closed curves with arbitrary big length. □

Proposition 7. *Let S be a compact convex surface and let R and $\overline{S \setminus R}$ be complementary regions in S. Then there is not an upper bound for the ratio*

$$\frac{d_M(R,S)}{P(R,S)}.$$

Proof. It is analogous to the proof of the Proposition 1. □

If we look for the lower bound of the ratio $\frac{d_M(R,S)}{P(R,S)}$ in the case of planar cuts, the first observation is that the plane providing the optimal case should pass through the center of symmetry, because any other plane would preserve the maximal diameter and decrease the relative perimeter.

A compactness argument would guarantee that this lower bound exists, and its value depends on the particular surface that we are considering. If the surface were a surface of revolution its determination would be a 1-parameter problem.

We can even provide a global lower estimate of this ratio:

Proposition 8. *Let S be a compact convex surface and let R and $\overline{S \setminus R}$ be complementary regions obtained dividing S by a plane Π. Then,*

$$\frac{d_M(R,S)}{P(R,S)} \geq \frac{1}{\pi}.$$

The equality is attained if S is a double disc and Π is the plane containing it.

Proof. Let C be the n–dimensional convex body bounded by S. Then, $d_M(R,S) \geq D(\Pi \cap C)$, and $P(R,S) = P(\Pi \cap C)$. The bound follows from the well-known inequality $\frac{D(K)}{P(K)} \geq \frac{1}{\pi}$ for planar convex bodies (see, for instance [14], [16]). □

Proposition 9. *Let S be a centrally symmetric compact convex surface and let α be a Jordan curve on S dividing S into two complementary regions, R and $\overline{S \setminus R}$ such that $A(R) = A(S)/2$. Then,*

$$\frac{d_M(R,S)}{P(R,S)} \leq \frac{D(S)}{2\delta_m(S)}.$$

This bound is not always attained. It is attained in the case that the distance between any pair of antipodal points is constant.

Proof. As α divides S into two regions of equal area; Lemma 1 guarantees that α contains two antipodal points p and p'. Then we have $P(R,S) \geq 2d_i(p,p') \geq 2\delta_m(S)$ and the inequality follows from $d_M(R,S) \leq D(S)$.

□

3. Minimum Relative Diameter

Proposition 10. *Let S be a compact convex surface and let R and $\overline{S \setminus R}$ be complementary regions in S. Then*

$$\frac{A(R,S)}{d_m(R,S)^2} \geq 0$$

and the bound is the best possible.

Proof. There are two points p, q such that $d_i(p,q) = D(S)$. Let γ be the corresponding path on S of length $D(S)$ (a half–meridian). Define $R_\epsilon := \{x \in S : \exists y \in \gamma \text{ such that } d_i(x,y) < \epsilon\}$ as the geodesic tube with radius ϵ rounding a half-meridian.

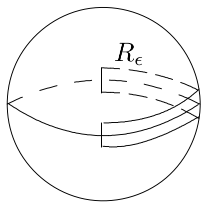

Figure 10

If ϵ goes to 0, we have that the surface area of R_ϵ goes to 0 and the minimum relative diameter to $D(S)$. Then the ratio goes to 0.

□

If we want to find the lower bound of the ratio $\frac{A(R,S)}{d_m(R,S)^2}$ in the case of planar cuts, a compactness argument would guarantee that this lower bound exists, and its value depends on the particular surface that we are considering.

As an example we compute this lower bound for the sphere:

Proposition 11. *Let $\mathbb{S}^2$ be the unit sphere; if α is the intersection of $\mathbb{S}^2$ with a plane, α divides $\mathbb{S}^2$ into two complementary regions, R and $\mathbb{S}^2 \setminus R$. Then,*

$$\frac{A(R,\mathbb{S}^2)}{d_m(R,\mathbb{S}^2)^2} \geq \frac{2}{\pi}.$$

The equality is attained only if R is a half-sphere.

Proof. Let R the region of $\mathbb{S}^2$ obtained subdividing $\mathbb{S}^2$ with a plane Π. Computing the relative area and the minimum relative diameter we obtain:

$$A(R, \mathbb{S}^2) = 2\pi(1 - \cos\varphi) \text{ and } d_m(R, \mathbb{S}^2) = 2\varphi,$$

where $\varphi \in (0, \pi/2]$ is the angle between the axis of the sphere perpendicular to Π and the straight line segment determined by the center of the sphere and any point of ∂R. (Fig. 11)

Then,

$$\frac{A(R, \mathbb{S}^2)}{d_m(R, \mathbb{S}^2)^2} = \frac{\pi(1 - \cos\varphi)}{2\varphi^2}.$$

This is a decreasing function with respect to φ, so the minimum is attained when $\varphi = \pi/2$:

$$\frac{A(R, \mathbb{S}^2)}{d_m(R, \mathbb{S}^2)^2} \geq \frac{2}{\pi}.$$

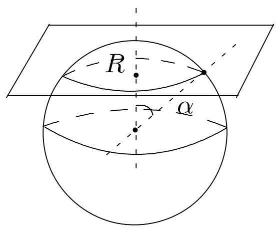

Figure 11

□

We have obtained a global lower estimate of this ratio:

Proposition 12. *Let S be a compact convex surface and let R and $\overline{S \setminus R}$ be complementary regions obtained dividing S by a plane Π. Then,*

$$\frac{A(R, S)}{d_m(R, S)^2} \geq 0,$$

and equality is attained, for instance, for the cylinder and for other non strictly convex surfaces.

Proof. Let $\mathbb{C}$ be a cylinder generated by a straight line segment l. Let $\{\Pi_i\}$ be a sequence of planes intersecting $\mathbb{C}$ and parallel to l such that the distance of Π_i to the axis of revolution is $1 - 1/i$. These planes determine a sequence of subsets of $\mathbb{C}$, $\{R_i\}$ such that $A(R_i, \mathbb{C})$ goes to 0 and $d_m(R_i, \mathbb{C})$ goes to c (length of l) when $i \to \infty$ (Fig. 12). Then,

$$\lim_{i \to \infty} \frac{A(R_i, \mathbb{C})}{d_m(R_i, \mathbb{C})^2} = 0.$$

□

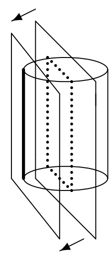

FIGURE 12

If we consider the fencing problem case, we obtain the following result:

Proposition 13. *Let S be a compact convex surface. Let α be a Jordan curve on S dividing S into two complementary regions of equal area, R and $\overline{S \setminus R}$. Then,*

$$\frac{A(R,S)}{d_m(R,S)^2} \geq \frac{A(S)}{2D(S)^2}.$$

The proof is analogous to the proof of Proposition 2.

Now, we are going to study the ratio between the minimum relative diameter and the relative perimeter.

Proposition 14. *Let S be a compact convex surface and let α be a Jordan curve on S dividing S into two complementary regions, R and $\overline{S \setminus R}$. Then*

$$\frac{d_m(R,S)}{P(R,S)} \geq 0$$

and the bound is the best possible.

Proof. It is sufficient to consider simple closed curves with arbitrary big length. $\square$

Remark 3. *If we want to compute the lower bound of this ratio in the case of planar cuts, using a similar argument as in Proposition 8, we obtain:*

$$\frac{d_m(R,S)}{P(R,S)} \geq \frac{1}{\pi},$$

where the equality is attained if S is a double disc and Π is a plane containing it.

Proposition 15. *Let S be a compact convex surface and let α be a Jordan curve on S dividing S into two complementary regions, R and $\overline{S \setminus R}$. Then there is no general upper bound for the ratio*

$$\frac{d_m(R,S)}{P(R,S)}.$$

Proof. There are surfaces for which this ratio is arbitrarily large; consider for instance the ellipsoid of revolution $\mathbb{E}^2[(a,a,b)] := \{(x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, 0 < a < b\}$. If b tends to infinity,

$$\frac{d_m(R, \mathbb{E}^2[(a,a,b)])}{P(R, \mathbb{E}^2[(a,a,b)])}$$

attains an arbitrary large value if R is bounded by the circle $\mathbb{E}^2[(a,a,b)] \cap \{z = 0\}$.

$\square$

In the particular case of the sphere, we can prove the following result:

Proposition 16. *Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere. Let R and $\overline{\mathbb{S}^2 \setminus R}$ be two regions from $\mathbb{S}^2$ obtained subdividing $\mathbb{S}^2$ by a simple closed curve. Then,*

$$\frac{d_m(R, \mathbb{S}^2)}{P(R, \mathbb{S}^2)} \le \frac{1}{2},$$

and the equality is attained when R is bounded by two half-meridians.

Proof. There are points $a, a' \in R$ and $b, b' \in \overline{\mathbb{S}^2 \setminus R}$ such that $d_i(a, a') = D(R)$ and $d_i(b, b') = D(\overline{\mathbb{S}^2 \setminus R})$. We distinguish two cases:

(1) If $a, a' \in \partial R$ (or $b, b' \in \partial R$), then $P(R, \mathbb{S}^2) \ge 2d_i(a, a') \ge 2d_m(R, \mathbb{S}^2)$.

(2) If this is not the case for both pairs then at least one of each pair is contained in the interior. Let us assume that $a \in \mathrm{int}(R)$, $b \in \mathrm{int}(\overline{\mathbb{S}^2 \setminus R})$. Because of their definition, a' and b' are necessarily antipodal to a and b, respectively (else the distance could be increased along the great circle containing a, a' and b, b', respectively). In particular, $d_m(R, \mathbb{S}^2) = \pi = d_M(R, \mathbb{S}^2)$. By appropriate translations on the sphere we can further assume that $a' \in \partial R$ and $b' \in \partial R$. Now we consider the curve C antipodal to ∂R. It connects a with b, which are contained in different components of $\mathbb{S}^2 \setminus \partial R$. By continuity C intersects ∂R and hence ∂R contains two antipodal points and so $P(R, \mathbb{S}^2) \ge 2\pi = d_m(R, \mathbb{S}^2)$.

$\square$

References

[1] Bernstein, F.: Über die isoperimetrische Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene, *Math. Ann.* **60** (1905), 117-136.

[2] Burago, D., Burago, Y. and Ivanov, S.: A Course in Metric Geometry, *American Mathematical Society (publisher)*, (1984).

[3] Bieberbach, A.: Über eine Extremaleigenschaft des Kreises, *Jber. Deutsch. Math.-Vereinig* **24** (1915), 247–250.

[4] Cerdán, A., Schnell, U. and Segura Gomis, S.: On relative geometric inequalities, *Math. Ineq. and Appl.* **Vol. 7 Num. 1**, (2004), 135–148

[5] Cerdán, A., Miori, C., and Segura Gomis, S.: On relative isodiametric inequalities, *Beitrage zur Algebra und Geometrie*,**Vol. 45 Num. 2**,(2004), 595–605.

[6] Cerdán, A.: Comparing the relative volume with the relative inradius and the relative width, *to appear in Journal of Ineq. and Appl.*, (2006).

[7] Cianchi, A.: On relative isoperimetric inequalities in the plane, *Boll. Unione Mat. Italiana,* **7 3-B** (1989), 289-325.

[8] Cotton , A., Freeman D., Gnepp, A., Ng, T., and Yoder, C.: The isoperimetric problem on some singular surfaces, *Journal of the Australian Mathematical Society* **78:2** (April 2005) 167-199.

[9] Gromov, M.: Metric structures for Riemannian and non-Riemannian spaces, *Springer*, Berlin (2007).

[10] Jost, J.: Riemannian Geometry and Geometric Analysis, *Springer-Verlag*, Berlin (2002)

[11] Markvorsen, S.: On curvature and shape: a comparison geometric survey, *Exposition. Math.* **13** (5) (1995), 417–432.

[12] Osserman, R.: The isoperimetric inequality, *Bulletin American Mathematical Society* **6** (1978), 1182–1238.

[13] Peri, C.: On relative isoperimetric inequalities, *Conferenze del Seminario di Matematica dell'Università di Bari*, **279**, 2000.

[14] Santaló, L.: Sobre los sistemas completos de desigualdades entre tres elementos de una figura convexa plana, *Math. Notae* **17** (1961), 82-104.

[15] Schneider, R.: Convex bodies: the Brunn-Minkowski theory, *Cambridge University Press*, Cambridge, (1993).

[16] Scott, P.R. and Awyong, P.W.: Inequalities for Planar Convex Sets, *J. Ineq. Pure and Appl. Math.* **1**(1), Art. 6 (2000), .