Chapter 1

Introduction

The Lie group analysis of differential equations was developed by Sophus Lie during his work at Leipzig University in 1873-1874 and later in the end of the 19th - beginning of the 20th century. Sophus Lie was deeply unsatisfied with the theory of differential equations to that time. The textbooks concerning differential equations were to some extent similar to cook books: take this substitution, apply it to this equation and you get the solution or obtain a reduction of the equation to a simpler one. Sophus Lie decided to find an internal structure which could allow one to uncover the procedure of discovering all these substitutions, if they exist. His ideas on point transformations and their algebraic structure are represented in the books [52], [53], [54]. His books are written in a clear and understandable way. Sophus Lie was a Norwegian but he used German language brilliantly, though it was not his mother tongue.

Sophus Lie provided the complete classification of all ordinary differential equations (ODEs) up to the third order. Additionally he studied the group structure of two dimensional linear partial differential equations (PDEs) in detail. The quantity and quality of Lie's work was so impressive and powerful, that for many years most mathematicians assumed that there was no room any more for a further development of ODEs or PDEs within the context of the Lie group analysis. The main focus was placed upon the development of the theory of Lie algebras, Lie groups and later on to the quantum-group theory.

The idea to use the Lie group analysis to study the PDEs came back into practice during the 1960's, with a development of new models in Mathematical Physics. Using the modern achievements in the theory of Lie algebras and Lie groups it was possible to get new results and renovate the theory of Lie group analysis of differential equations.

The group of Ovsiannikov [61] and later Ibragimov [41] played the leading role in this process. In the last decades of the 20th century this method was widely used in Mathematical Physics. Many books devoted to the Lie group theory with applications in Physics and Mathematical Physics were written in this time. There are textbooks for beginners like [39], [20], [28] and more sophisticated ones like [60],[11], [9], [10]. This topic was included in education curriculae for mathematics and physics students in many universities over the wold.

Financial Mathematics rapidly developed in the 20th century. Since the famous works of Black, Scholes and Merton's [8], [56] Financial Mathematics is moving from pure econometric and stochastic methods to the more and more complicated areas of Mathematics. The main subject of the Black-Scholes (BS) model was a linear parabolic PDE. In the framework of this model it was possible to get directly the simple and useful solutions in an exact analytic form [8]. In the language of the Lie group analysis the famous Black-Scholes (BS) formula represent an invariant solution of this equation. With the work of Black and Scholes the rich arsenal of methods of theory of PDE was suddenly available for the financial mathematicians. The properties of parabolic PDEs were very well studied before in the framework of physics, one just needed to reformulate and interpret the results for the new application.

This circumstances not only made the BS model famous and the authors the Nobel price winners, but also allowed to use this model in financial companies. On the other hand it attracted a lot of specialists in the area of PDEs with their own instruments and experience in the PDE theory. The BS model was developed for an ideal financial market, it was transparent and easy to use. However different improvements are necessary if we take into account deviations of a real market from the ideal one. A lot of models given in the form of nonlinear PDEs which take into account different frictions on the financial market appeared in the last 40 years.

After financial crises in 2008 the practitioners all over the world see that illiquidity is one of the most demanding problems at the moment. Financial Mathematics applied in the industry was mainly focused on applications of statistical methods to the models implied the linearity of the corresponding processes. Yet it is known that the modeling of hedging and pricing options in non-ideal markets with illiquidity problems or transaction costs leads to strong nonlinearities in PSDEs and correspondingly PDEs. In many cases it is impossible to regard these PDEs simply as a perturbation of a linear approximation. Moreover, known numerical methods may break down due to the strong non-linearity. Because of that, Financial Mathematics demands new mathematical methods to work with these new phenomena. Hence, one needs to apply a broader spectrum of the advanced mathematical methods to nonlinear equations. One of the methods is the method of geometrical analysis, especially Lie group analysis. This method is new in this developing area and so far, the Lie group analysis applied to financial models is not reflected at all in the textbooks.

Only 25 years after the famous paper of Black and Scholes the first paper [34] was written where the Lie group analysis method was applied to the BS model and to similar PDEs appearing in Financial Mathematics. A bit later the group of Leach started to use this method in applications to linear and quasilinear PDE models in Financial Mathematics [58]. We also used this method to study strongly nonlinear models in [15], [12] [19], [16],[13],[17] [14],[18].

Some of these models and results of their analyses are presented in Chapter 9. Nowadays one can see a growing interest to the Lie group analysis and more and more papers appear which use this method in applications to new models in Financial Mathematics. On one hand the Lie algebraic structure of the studied equations is used to get invariant solutions and to use them as benchmarks for instance. On the other hand this structure is used to develop new numeric schemes [26], [27], [7] which are more efficient and stable [67], [65].

This text book is suitable for the students of mathematical departments on the master level or the higher bachelor level who are interested in applications of modern mathematical methods in Financial Mathematics, Mathematical Economics or Applied Mathematics with the focus on economic aspects. As prerequisite courses reader needs usual bachelor level courses in ODEs (a lecture course on theory of PDEs would be rather handy as well). In comparison to the well known text books devoted to the Lie group analysis applied to differential equations we do not use any physical intuition. The most of these text books were written when the Lie group analysis was mainly addressing the needs of Mathematical physics and such analogies were very welcome. Nowadays students who are interested in economics or Financial Mathematics do not have any physics-background and are not able to use such hints linked to physical phenomena. The textbook with a number of examples will be useful not only for students who are interested in Financial Mathematics but also for people who are working in other areas of research that are not directly connected with Physics (for instance in such areas of Applied Mathematics like mathematical economy, bio systems, coding theory and etc.).

The book has the following structure.

In Chapter 2 and Chapter 3 the ideas of point transformations on $\mathbb{R}^2$, one-parameter group and invariants are introduced on the intuitive level with a large amount of examples, problems which are accompanied with very detailed solutions and tasks to solve on their own are presented. In the similar way in Chapter 4 the idea of first order ODEs, Lie point symmetries and solutions of these ODEs using their internal algebraic structure is introduced. Also Chapter 4 contains a lot of examples, problems with extensively described solutions that allow students to get experienced with this material. The next two chapters, Chapter 5 and Chapter 6, have a more technical nature. In Chapter 5 the idea of a prolongation is introduced. In order to apply Lie group analysis to higher oder differential equations one needs to be familiar with the concept of the prolongation. In Chapter 6 some properties of Lie algebras are introduced and explained. These properties are used in the subsequent chapters to describe the different families of invariant solutions. A lot examples, detailed problems accompanied by the solutions and tasks to solve are included.

Chapter 7 is devoted to Lie group analysis of higher order ODEs. This chapter is an important step towards the Lie group analysis in application to Financial Mathematics. Since new models in Financial Mathematics are often represented by nonlinear PDEs, after using a reduction procedure they are usually transformed to ODEs. So, it is important that the reader is familiar with the equations of such type and can work with them.

The technical tools are introduced and adjusted to the needs of Lie group analysis of PDEs in Chapter 8. The result of application of the Lie group analysis is demonstrated on two examples. For two linear PDEs, the heat equation and the Black-Scholes equation, we provide admitted Lie algebras, describe the symmetry group structure and discuss the idea of invariant solutions. It is shown that both equations have isomorphic and similar Lie algebras and may be reduced to each other with simple transformations. These transformations are often used in different papers or even textbooks but one rarely mentions their nature and background concepts that stand behind this similarity. In this chapter we obtain an explicit analytical solution for a Call option that was first presented in the work of Black and Scholes as an invariant solution of this PDE.

After the technical tools are developed in previous chapters, in Chapter 9 continue to study different models describing processes in financial markets under frictions with Lie group analysis. In the beginning we give some short introduction in the theory of financial markets, define the ideal market and list the assumptions under which the BS model is valid. Further we discuss how this model can be improved and list some modern models. These models were developed under different economical and financial assumptions and the corresponding PDEs looks also rather different. But after Lie group can prove that the models are often connected to each other and have isomorphic algebraic structures.

We provide a lot of examples, problems with detailed solutions and some task to solve so that the readers can always check their level of understanding. In Appendix we present some optimal systems of subalgebras for Lie algebras studied in Chapter 9 and a formulation of Ito Lemma. We also provide an example of usage of software package **IntroToSymmetry** which is an additional package to **Mathematica** computer algebra package. The book contains a list of references with the most important monographs and textbooks as well as some papers which are directly connected to the book material. The short subject index makes the book more convenient.